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QUASISTATIONARY FINITE-INTERVAL CONTROL OF THE MOTION OF HYBRID OSCILLATORY SYSTEMS*

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A study of the uniaxial (translational or rotational) motion of a mechanical system containing perfectly rigid bodies (material points or flywheels) is presented. These parts of the system are connected in series by elastic elements with distributed characteristics (springs, bars or shafts); point controls (forces or torques) are applied to the rigid bodies or at the ends of the elastic connectors. An approximate solution is proposed to the problem of steering a hybrid (discrete-continuous) system to a desired state as a whole, without relative elastic vibrations. Underlying the constructive approach of this paper is the quasistationary nature of forced elastic displacements, which is known to be valid provided the controls are sufficiently smooth and even.

1. ***Statement of the problem.*** To fix our ideas, we shall first consider unidirectional controlled motion (along the X axis) of a simple hybrid vibrating system (Fig.1). The mechanical system contains a perfectly rigid body m (a material point) connected to an elastic bar (distributed spring) ol. The geometric, inertial and elastic characteristics of the elastic element are assumed to be constant: its length l, linear density ρ and compressional rigidity σ. The conditions of the motion are such that the elastic deformations can be dealt with in the linear theory.

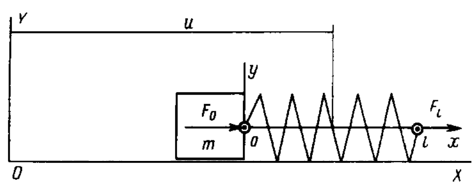

Fig.1

Variable point controlling forces are applied to the absolutely rigid body m (at $x = 0$) and to the end of the elastic element $(x = l)$; we will denote them by $F_0(t)$ and $F_l(t)$, respectively. One of the functions $F_{0,l}(t)$ may vanish identically. The initial state of the system (at $t = 0$) is assumed to be known. The problem is to bring the system at some finite time $t = T < \infty$ to a specified state of motion, by suitable choice of the controls $F_{0,l}(t)$ from some admissible class. In applied problems one usually imposes the basic condition that there be no elastic vibrations at times $t > T$ (see /1-5/ etc.).

We will now consider the mathematical formulation of this problem: to determine the dynamics and controlled motion of a hybrid system driven by point controls through the boundary. The equations of state of the distributed elastic system and boundary conditions for the forces are

$$\rho u^{\bullet\bullet} = \sigma u'', \quad 0 < x < l; \quad |u = u(t, x) \tag{1.1}$$
$$mu^{\bullet\bullet}(t, 0) = \sigma u'(t, 0) + F_0(t), \quad \sigma u'(t, l) = F_l(t)$$

Here $u = u(t, x)$ is the absolute displacement of the section $x, x \in [0, l]$ at time t. Dots and primes denote differentiation with respect to t and x, respectively. The motion of system (1.1) is considered over some bounded interval of time $t, t \in [0, T_f], T \leqslant T_f < \infty$. The initial state of the system, i.e., the displacements and velocities of the sections of the elastic element and of the rigid body are given:

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$$u(0, x) = g^0(x), \quad u^{\bullet}(0, x) = h^0(x), \quad x \in [0, l] \tag{1.2}$$

where g^0 and h^0 are sufficiently smooth functions with $g^0(+0) = g^0(0)$, $h^0(+0) = h^0(0)$. The control problem is to select admissible compactly supported /1/ functions $F_{0,l}(t)$, $F_{0,l}(t) \in \{F(t)\}$, $t \in [0, T_f]$ such that

$$u(T, x) = g^T(x), \quad u^{\cdot}(T, x) = h^T(x), \quad x \in [0, l] \tag{1.3}$$

The functions $F_{0,l}(t)$, $g^{0,T}(x)$, $h^{0,T}(x)$ are to be chosen so that a classical or strong solution of the problem exists /1-3, 6, 7/. We note that when $u(t, x) \equiv s(t)$ the system moves as a whole without relative elastic vibrations. Thus, if we require that $g^T = \text{const}, h^T = \text{const}$, then if $F_{0,l}(t) \equiv 0$ at $t \geqslant T$ the system will move as a whole at a constant velocity h^T (see below).

It will be more convenient to solve the boundary-value problem if we non-dimensionalize the variables and parameters in system (1.1)-(1.3). This will also enable us to reduce the number of parameters. We define the dimensionless variables as follows:

$$t^* = \nu t, \quad x^* = xl^{-1}, \quad m^* = m(\rho l)^{-1}, \quad \nu^2 = \sigma(\rho l^2)^{-1} \tag{1.4}$$
$$u^*(t^*, x^*) \equiv u(t^*, \nu^{-1}, x^* l)\, l^{-1}, \quad f_{0,1}(t^*) \equiv F_{0,l}(t^*\nu^{-1})(\rho l^2\nu^2)^{-1}$$
$$g^{0,T*}(x^*) \equiv g^{0,T}(x^* l)\, l^{-1}, \quad h^{0,T*}(x^*) \equiv h^{0,T}(x^* l)(l\nu)^{-1}$$

Henceforth the asterisk will be omitted. The units of time, length and mass in (1.4) are related to the corresponding parameters of the elastic part of the system, which is assumed to be significant. The units of measurement could also have been defined otherwise; how this should be done in each specific case is dictated by considerations of convenience. At any rate, the transformations (1.4) reduce the equations of motion to the form (1.1) with $\rho = \sigma = l = 1$; the intervals of time are also transformed accordingly ($\nu T \to T$, $\nu T_f \to T_f$).

A good mechanical model of system (1.1)-(1.3) is a perfectly rigid flywheel mounted on an elastic shaft with distributed torsional rigidity and inertia parameters. The model of a stretched string with a material point attached at one of its ends leads to analogous equations; and there are other possible mechanical interpretations of problem (1.1)-(1.3). The case $m = 0$ (the control of a bar) was considered by the present author in /8/, where methods of the moment problem /1/ were used to construct an exact solution of the control problem (1.1)-(1.3), which optimizes a mean-square cost function, in terms of the initial arbitrary functions $g^{0,T}(x)$, $h^{0,T}(x)$.

2. *Construction of the solution for given applied forces.* If the functions $f_{0,1}(t)$, $t \in [0, T_f]$, are known, a solution of the boundary-value problem (1.1) with initial data (1.2) is constructed by separation of variables (the Fourier method /1-3, 6, 7/). The corresponding selfadjoint boundary-value problem and its solutions - the eigenvalues and eigenfunctions of the problem - are as follows:

$$X'' + \lambda^2 X = 0, \quad X'(0) = mX''(0) = -m\lambda^2 X(0), \; X'(1) = 0 \tag{2.1}$$
$$D(m, \lambda) \equiv \lambda^2(\sin\lambda + m\lambda\cos\lambda) = 0, \quad \lambda = \arg D$$
$$\lambda_0 = 0, \quad \lambda_n = \lambda_n(m), \quad n = 1, 2, \ldots$$
$$X_0(x, m) = a_0 = \text{const}, \quad X_n(x, m) = a_n\varphi_n(x, m)$$
$$\varphi_n(x, m) \equiv \cos\lambda_n x - m\lambda_n\sin\lambda_n x \quad (\varphi_0 \equiv 1)$$

Computed eigenvalues $\lambda_n(m), n \geqslant 1$ (the roots of the characteristic equation $D(m, \lambda) = 0$) for different $m \geqslant 0$ and $n = 1, 2, \ldots$ may be found, e.g., in /9/ and elsewhere. It follows from (2.1) that for fixed $m > 0$ the asymptotic behaviour of the eigenvalues $n \gg 1$ is given by

$$\lambda_n(m) = \lambda_n^0 - (m\lambda_n^0)^{-1} + O((m\lambda_n^0)^{-2}), \quad \lambda_n^0 = {}^1/_2\pi(2n - 1)$$

This asymptotic formula also holds for all $n \geqslant 1$ and $m \gg 1$; in the limit as $m \to \infty$ we obtain the case of a bar fixed at its left end, for which the eigenvalues and orthonormal eigenfunctions are $\lambda_n = \lambda_n^0$, $\xi_n = \sqrt{2}\sin\lambda_n x$.

Now let us consider the asymptotic behaviour of the system for small m, $0 \leqslant m \ll 1$. If $m > 0$, this can be derived for "not too large" $n, n = 1, 2, \ldots, n^*$, where $mn^* \ll 1$; we have

$$\lambda_n(m) = \pi n - m(\pi n)(1 - m) + O((m\pi n)^3)$$

In the limit of $m = 0$ we obtain the case of a free homogenous bar, for which the eigenvalues and orthonormal eigenfunctions are /8/:

$$\lambda_0 = 0, \quad \xi_0 \equiv 1; \quad \lambda_n = \pi n, \quad \xi_n = \sqrt{2}\cos\lambda_n x$$

Note that if $m \gg 1$ $(m \to \infty)$ and $f_0 \sim m$ the boundary condition (1.1) at $x = 0$ takes the form of a kinematic boundary control; $mu^{\cdot\cdot}(t, 0) = f_0(t)$. This clearly means that the mass m and left end of the bar are displaced independently of its elastic vibrations. The study of this control problem is of some practical interest.

The system of eigenfunctions $\{X_n(x)\}$, $n \geqslant 0$, is orthogonal with weight $\chi(x, m) = 1 + m\delta(x)$, where $\delta(x)$ is the Dirac delta-function /7/. This is proved by direct integration of the expression $X_nX_{n'}\chi$ over x, $x \in [0, 1]$, subject to conditions (2.1). The eigenfunctions $\xi_n(x, m)$, orthonormalized with respect to χ, form a basis $\{\xi_n(x, m)\}_\chi$ $(n = 0, 1, \ldots)$:

$$\xi_n(x, m) = X_n(x, m)\,\|X_n\|_\chi^{-1} = \varphi_n(x, m)\,r_n^{-1}(m) \tag{2.2}$$

$$(\xi_n, \xi_{n'})_\chi = \delta_{nn'}; \quad n, n' = 0, 1, 2, \ldots$$

$$r_n^2(m) = \|\varphi_n\|_\chi^2 \equiv \int_0^1 \varphi_n^2(x, m)\,\chi(x, m)\,dx > 0$$

$$r_0^2 = 1 + m, \quad r_n^2 = {}^1/_2 + m + {}^1/_2 m^2\lambda_n^2 - (1 + m^2\lambda_n^2)(2m\lambda_n^2)^{-1}\sin^2\lambda_n$$

$$(n \geqslant 1)$$

Here $\delta_{nn'}$ is the Kronecker delta; in the evaluation of $r_n^2(m)$ $(n = 1, 2, \ldots)$ we have used the identity $\sin\lambda_n + m\lambda_n\cos\lambda_n \equiv 0$, since $\lambda_n(m)$ are the roots of the characteristic Eq.(2.1). The above expressions are obtained by letting $m \to 0$ or $m \to \infty$.

Using a well-known method /10/ (see also /8/), we obtain a denumerable system of differential equations for the Fourier coefficients $\theta_n(t)$ $n \geqslant 0$ of the expansion of the desired solution of problem (1.1), (1.2); these equations are connected through the controls $f_{0,1}(t)$:

$$u(t, x) = \sum_{n=0}^{\infty} \theta_n(t)\,\xi_n(x, m) \equiv \vartheta_0(t) + \sum_{n=1}^{\infty} \theta_n(t)\,\xi_n(x, m) \tag{2.3}$$

$$\vartheta_0^{\cdot\cdot} = \varphi_0(t), \quad \varphi_0(t) \equiv [f_0(t) + f_1(t)]\,M^{-1}, \quad \vartheta_0(0) = s^0, \quad \vartheta_0^{\cdot}(0) = v^0$$

$$\theta_n^{\cdot\cdot} + \lambda_n^2\theta_n = \varphi_n(t), \quad \varphi_n(t) \equiv \xi_n(0, m)\,f_0(t) + \xi_n(1, m)\,f_1(t)$$

$$\xi_n(0, m) = r_n^{-1}, \quad \xi_n(1, m) = (1 + m^2\lambda_n^2)\,r_n^{-1}\cos\lambda_n$$

$$\theta_n(0) = g_n^0, \quad \theta_n^{\cdot}(0) = h_n^0, \quad n \geqslant 1$$

The variable $\vartheta_0 = \theta_0 r_0^{-1}$ represents the coordinate of the centre of mass of the system; $M = (1 + m)$ is the total mass and $(f_0 + f_1)$ the total force. The remaining coefficients θ_n $(n \geqslant 1)$ describe vibrations in all modes relative to the moving centre of mass; if $\theta_n(t) \equiv 0$ $(n \geqslant 1)$, system (1.1) is moving rigidly (see Sect.1). The initial values g_n^0, h_n^0 $(n \geqslant 0)$ are the Fourier coefficients with weight χ of the initial distribution functions $g^0(x)$, $h^0(x)$ (1.2) in terms of the basis $\{\xi_n(x, m)\}_\chi$, and $s^0 = g_0^0 r_0^{-1}$, $v^0 = h_0^0 r_0^{-1}$ are the initial position and velocity of the centre of mass.

Interestingly enough, if we assume that for some $n \geqslant 1$ the functions $f_0(t)$ and $f_1(t)$ are proportional,

$$f_0(t) \equiv -\cos\lambda_n\,(1 + m^2\lambda_n^2)\,f_1(t), \quad t \in [0, T]$$

then the controls will have no effect on the mode in question, since by (2.3) $f_n(t) \equiv 0$. Vibrations in this mode will be free; in particular, there will no none at all if $g_n^0 = h_n^0 = 0$.

The motion of the denumerable-dimensional system (2.3), for given controls $f_0(t), f_1(t), t \in [0, T]$, is determined by quadratures:

$$\vartheta_0(t) = s^0 + v^0 t + \int_0^t (t - \tau)\,\varphi_0(\tau)\,d\tau, \quad \vartheta_0^{\cdot}(t) = v^0 + \int_0^t \varphi_0(\tau)\,d\tau \tag{2.4}$$

$$\theta_n(t) = g_n^0\cos\lambda_n t + \frac{h_n^0}{\lambda_n}\sin\lambda_n t + \frac{1}{\lambda_n}\int_0^t \sin\lambda_n(t - \tau)\,\varphi_n(\tau)\,d\tau$$

$$\theta_n^{\cdot}(t) = d\theta_n(t)/dt, \quad n \geqslant 1 \quad g_n^0 = (g^0, \xi_n)_\chi, \quad h_n^0 = (h^0, \xi_n)_\chi$$

We shall assume that the coefficients g_n^0, h_n^0 decrease fairly rapidly as n increases, i.e., the functions $g^0(x)$, $h^0(x)$ are chosen from the appropriate smoothness class /2, 6-8/. The boundary controls $f_{0,1}(t)$ are also assumed to be sufficiently smooth. Under these conditions, substitution of (2.4) into the series (2.3) produces the desired strong or classical solution $u = u(t, x)$ of problem (1.1), (1.2). This series, and the series for the derivatives of u with respect to t, x, obtained by term-by-term differentiation, will either converge in the mean square or uniformly or will lead to generalized functions (if the smoothness conditions are not satisfied).

3. Approximate solution of the simplified fundamental problem. System (2.3) can be interpreted as a denumerable collection of pendulums (linear oscillators) placed on a common moving base, with the latter subjected to controlling forces $f_{0,1}(t)$ /8,11/. In general the frequencies at which the pendulums oscillate are incommensurate, i.e., the free vibrations of the system are almost periodic. The control problem for the motion (1.1)-(1.3) with general terminal conditions $g_n^T, h_n^T \neq 0$, $g_n^T = \theta_n(T)$, $h_n^T = \theta_n^{\cdot}(T)$, based on system (2.3), can be approximately asymptotically solved for $T \gg 1$ $(T \to \infty)$ in the formulation suggested in /1/. However, his approach to the control problem involves the realization of an extremely complicated function, involving a polynomial and an almost periodic function of t. Moreover, the control cannot always be implemented to within arbitrary prescribed accuracy in powers of the small parameter, whose value is related to the length T of the time interval and the magnitude of the control.

We propose another approach to the approximate solution of the control problem to within the desired accuracy for the basic, bounded formulation. To be specific: let us assume that $h_n^0 = g_n^0 = 0$, $n \geqslant 1$, i.e., there are initially no relative vibrations in the system. They may be suppressed at a preliminary phase of the control process or damped out by a small natural dissipation. The problem is now to bring the object to the desired state of motion as a whole; moreover, at the end of the process the elastic vibrations must be suppressed to within a prescribed degree of accuracy with respect to the small parameter ε $(T \sim \varepsilon^{-1})$:

$$|\theta_n(t)| = \frac{1}{\lambda_n}\left|\int_0^T \sin\lambda_n(t-\tau)\varphi_n(\tau)\,d\tau\right| \leqslant c_n\varepsilon^\eta \tag{3.1}$$

$$|\theta_n^{\cdot}(t)| = \left|\int_0^T \cos\lambda_n(t-\tau)\varphi_n(\tau)\,d\tau\right| \leqslant d_n\varepsilon^\eta$$

$$\varepsilon \in (0, \varepsilon_0],\ \varepsilon_0 \ll 1,\ \eta > 0,\ n \geqslant 1;\ f_{0,1}(t) \equiv 0,\ t > T$$

where c_n and d_n are coefficients that tend to zero fairly rapidly as $n \to \infty$.

We propose to seek the control functions $f_0(t), f_1(t)$ in the class of smooth, slowly varying and compactly supported functions /1/. In formal terms:

$$f_{0,1}(t) \equiv f_{0,1}(\varkappa) \in \Phi^K[0,\Theta],\quad \varkappa = \varepsilon t \in [0,\Theta],\quad T = \Theta\varepsilon^{-1} \tag{3.2}$$

$$\Phi^K[0,\Theta] = \{\varphi(\varkappa,\Theta):\quad \varphi(\varkappa) \not\equiv 0,\quad \varkappa \in [0,\Theta];\quad \varphi^{(k)}(\varkappa,\Theta) \equiv 0,$$

$$\varkappa \,\overline{\in}\, [0,\Theta];\quad \varphi^{(k)}(\varkappa,\Theta)\,|_{\varkappa=0,\Theta} = 0;\quad k = 0, 1, \ldots, K,\ K < \infty\}$$

Here $\{\varphi\}$ are functions of the slow time $\varkappa$ which vanish together with their derivatives of order up to K inclusive; the $(K+1)$-th derivative is assumed to be integrable in either the proper or the improper sense (see below); the coefficient $\Theta > 0$ is independent of ε.

The physical meaning of the assumption that $f_{0,1} = f_{0,1}(\varkappa)$ are slow functions is that in the duration T of the control process the elastic part of the system performs many (of the order of ε^{-1}) vibrations (in the lowest - first - mode); this is in fact usually the case in practice and does not constitute a prohibitive restriction. Indeed, usually $F_{0,l} = F_{0,l}(\Omega t)$, where Ω is the characteristic frequency of the control in dimensional time $t \in [0, T]$, for example, $\Omega \sim 1/T$. Non-dimensionalizing in accordance with (1.4), we get

$$f_{0,1}(\varepsilon t^*) \equiv F_{0,l}((\Omega/\nu)\,t^*)\,(\rho l^2\nu^2)^{-1}$$

where we have put $\Omega/\nu = \varepsilon \ll 1$.

Let us assume that the initial conditions (1.2) and terminal conditions (1.3) correspond to a state of motion of the object as a whole, without relative vibrations:

$$\vartheta_0(0) = s^0,\quad \vartheta_0^{\cdot}(0) = v^0 \tag{3.3}$$

$$\alpha^2[\vartheta_0(T) - s^T]^2 + \beta^2[\vartheta_0^{\cdot}(T) - v^T]^2 = 0,\quad \alpha^2 + \beta^2 > 0$$

$$g_n^{0,T} = h_n^{0,T} = 0,\quad n = 1, 2, \ldots$$

These conditions meant that $g^{0,T} = \text{const}$, $h^{0,T} = \text{const}$. The terminal condition (3.3) imposed on the motion of the centre of mass implies, if $\alpha^2 > 0$, $\beta^2 > 0$, that $\vartheta_0(T) = s^T$, $\vartheta_0^{\cdot}(T) = v^T$, where s^T, v^T are given. If $\alpha = 0$, the velocity $\vartheta_0^{\cdot}(T) = v^T$ is prescribed, while $\vartheta_0(T)$ is arbitrary; if $\beta = 0$ - vice versa. We may assume without loss of generality that $s^T = v^T = 0$; if not, we just apply one or both of the transformations $\vartheta_0 = s + s^T + v^T(t - T)$, $\vartheta_0^{\cdot} = v + v^T$ and $\vartheta_0 = s + s^T$, $\vartheta_0^{\cdot} = v + v^T$. In the first case we have a moving coordinate system $(s^{\cdot} = v^T)$, in the second $s^{\cdot} = v + v^T$.

We now consider the motion of the centre of mass of system (2.3) for control functions $f_{0,1}(\varkappa)$ of the form (3.2). If $\alpha, \beta \neq 0$, both of the following conditions must hold:

$$-\int_0^\Theta (\Theta - \varkappa)\varphi_0(\varkappa)\,d\varkappa = \varepsilon^2 s^0 + \varepsilon v^0 \Theta \quad (\alpha \neq 0) \tag{3.4}$$

$$-\int_0^\Theta \varphi_0(\varkappa)\,d\varkappa = \varepsilon v^0 \quad (\beta \neq 0)$$

but if $\beta = 0\ (\alpha \neq 0)$ or $\alpha = 0\ (\beta \neq 0)$ only one is necessary. By suitable choice of the functions $f_{0,1}(\varkappa) \in \Phi^K[0, \Theta]$ (3.2) one can make sure that conditions (3.4) are satisfied; where if $\varepsilon^2 |s^0| + \varepsilon |v^0| \sim 1$, i.e., $s^0 \sim \varepsilon^{-2}, v^0 \sim \varepsilon^{-1}$, then $\varphi_0(\varkappa) \sim 1$, but if $\varepsilon|s^0| + |v^0| \sim 1$, i.e., $s^0 \sim \varepsilon^{-1}, v^0 \sim 1$, then $\varphi_0(\varkappa) \sim \varepsilon$.

The second situation may prove preferable, since it leads to small $O(\varepsilon)$ elastic displacements, implying that linear elasticity theory is applicable to within the desired accuracy in ε. Other asymptotic forms for s^0, v^0 and $\varphi_0(\varkappa)$ for small ε, including intermediate forms, are also possible and admissible. The selection and construction of functions $f_{0,1}(\varkappa)$ in $\Phi^K[0, \Theta]$ involves no difficulties.. The form of the function $\varphi_0(\varkappa)$ for different K values is shown qualitatively in Fig.2, curves *1-3*. Curve *1* may have a singularity of the type $\varkappa^{1-\gamma}$, $0 < \gamma < 1$, $\varkappa \to 0$; this corresponds to $K = 0$, i.e., a function φ_0 which is almost a step function (on-off control) and yields almost time-optimal control. Curve *2* corresponds to $K = 1$; it is almost trapezoidal, with smoothed corners, and has finite angles of inclination. Curve *3* corresponds to $K \geqslant 2$. If one of conditions (3.4) is not imposed ($\beta = 0$ or $\alpha = 0$), the control $\varphi_0(\varkappa)$ may be picked in a form similar to that in Fig.2, but without any change in the sign of $\varphi_0(\varkappa)$, i.e., it will look like the part of $\varphi_0(\varkappa)$ over the interval $\varkappa \in [0, \Theta_0]$. Note that Θ may be fixed and assigned in advance, in which case the spread of values of $\varphi_0(\varkappa)$ will be determined by the initial conditions. Conversely, if the spread of values is bounded, e.g., $|\varphi_0(\varkappa)| \leqslant d_0$, the value of that parameter will be determined by solving the appropriate time-optimal control problem.

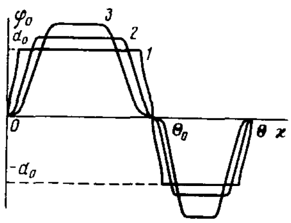

Fig.2

Thus, the controls $\varphi_0(\varkappa) \in \Phi^K[0, \Theta]$ enable one to change the motion of the centre of mass of the system, i.e., the variables $\vartheta_0, \vartheta_0^{\cdot}$ (or s, v), substantially. We shall show that when that is done elastic vibrations are induced relatively weakly for $t \in [0, T]$; we shall give estimates for the residual oscillations at $T < t \leqslant T_f$ in terms of the small parameter.

Elastic displacements during the control process, however, may be quite significant - of the order of the control $|f_0| + |f_1|$. Nevertheless, the displacements are quasistatic, as can be seen by evaluating $\theta_n(t), \theta_n^{\cdot}(t), n \geqslant 1$ by formulae (2.4) with $g_n^0 = h_n^0 = 0$.

We will first consider the control interval $t \in [0, T]$. Integrating formulae (2.4) successively by parts, we finally obtain

$$\theta_n(t) = \lambda_n^{-2}\varphi_n(\varkappa) - \varepsilon^2\lambda_n^{-4}\varphi_n''(\varkappa) + \varepsilon^4\lambda_n^{-6}\varphi_n^{(4)}(\varkappa) - \varepsilon^6\lambda_n^{-8}\varphi_n^{(6)}(\varkappa) + \varepsilon^8\lambda_n^{-10}\varphi_n^{(8)}(\varkappa) - \ldots + \Delta_{n,K}(\varkappa, \varepsilon) + \varepsilon^{K+1}\lambda_n^{-(K+2)}I_{n,K}(t, \varepsilon) \tag{3.5}$$

$$\theta_n^{\cdot}(t) = d\theta_n(t)/dt = \varepsilon\lambda_n^{-2}\varphi_n'(\varkappa) - \varepsilon^3\lambda_n^{-4}\varphi_n'''(\varkappa) + \varepsilon^5\lambda_n^{-6}\varphi_n^{(5)}(\varkappa) - \varepsilon^7\lambda_n^{-8}\varphi_n^{(7)}(\varkappa) + \varepsilon^9\lambda_n^{-10}\varphi_n^{(9)}(\varkappa) - \ldots + \Delta_{n,K}^*(\varkappa, \varepsilon) + \varepsilon^{K+1}\lambda_n^{-(K+1)}I_{n,K}^*(t, \varepsilon)$$

$$n \geqslant 1, \quad t \in [0, T], \quad T = \Theta\varepsilon^{-1}$$

The forms of the remainder terms $\Delta_{n,K}, \Delta_{n,K}^*$ and $I_{n,K}, I_{n,K}^*$ depend on the parity of K. For odd $K = 2k + 1\ (k = 0, 1, 2, \ldots)$,

$$\Delta_{n,K} = \pm\varepsilon^{K-1}\lambda_n^{-(K+1)}\varphi_{n(\varkappa)}^{(K-1)}, \quad \Delta_{n,K}^* = \pm\varepsilon^K\lambda_n^{-(K+1)}\varphi_n^{(K)}(\varkappa) \tag{3.6}$$

$$I_{n,K}(t, \varepsilon) = \mp\int_0^t \sin\lambda_n(t - \tau)\,\varphi_n^{(K+1)}(\varepsilon\tau)\,d\tau$$

$$I_{n,K}^*(t, \varepsilon) = \mp\int_0^t \cos\lambda_n(t - \tau)\,\varphi_n^{(K+1)}(\varepsilon\tau)\,d\tau$$

similarly, for even $K = 2k$,

$$\Delta_{n,K} = \pm\varepsilon^K\lambda_n^{-(K+2)}\varphi_n^{(K)}(\varkappa), \quad \Delta_{n,K}^* = \pm\varepsilon^{K-1}\lambda_n^{-(K+1)}\varphi_n^{(K-1)}(\varkappa) \tag{3.7}$$

$$I_{n,K}(t,\varepsilon) = \mp\int_0^t \cos\lambda_n(t-\tau)\,\varphi_n^{(K+1)}(\varepsilon\tau)\,d\tau$$

$$I_{n,K}^*(t,\varepsilon) = \pm\int_0^t \sin\lambda_n(t-\tau)\,\varphi_n^{(K+1)}(\varepsilon\tau)\,d\tau$$

The signs $\pm$ and $\mp$ in (3.6), (3.7) depend on the remainders of K and $K-1$ modulo 4. It is assumed (see (3.2) and the definition of the class Φ^K) that the integrals $I_{n,K}$ and $I_{n,K}^*$ (3.6), (3.7) exist for $t \in [0, \Theta\varepsilon^{-1}]$ and satisfy the following estimates in terms of ε (see, e.g., /4/):

$$|I_{n,K}|, \quad |I_{n,K}^*| = O(\varepsilon^{-\gamma}), \quad \gamma < 1, \quad |f_0| + |f_1| = O(1) \tag{3.8}$$

$$|I_{n,K}|, \quad |I_{n,K}^*| = O(\varepsilon^{1-\gamma}), \quad \gamma < 1, \quad |f_0| + |f_1| = O(\varepsilon)$$

Such estimates hold if the functions $\varphi^{(K+1)}(\varkappa)$ have a finite number of singularities of the type $(\varkappa - \varkappa_i)^{-\gamma}$, where i is the number of the integrable discontinuity of the second kind $(i = 1, 2, \ldots, i^*)$ /4/. Thus, in cases *1* and *2* of the curves $\varphi_0(\varkappa)$ shown in Fig.2 we obtain, respectively:

$$K = 0\colon\ \theta_n(t) = \lambda_n^{-2}\varphi_n(\varkappa) + \varepsilon O(I_{n,0}\lambda_n^{-2}) \tag{3.9}$$

$$\theta_n^{\boldsymbol\cdot}(t) = \varepsilon O(I_{n,0}^*\lambda_n^{-1}), \quad n \geqslant 1$$

$$K = 1\colon\ \theta_n(t) = \lambda_n^{-2}\varphi_n(\varkappa) + \varepsilon^2 O(I_{n,1}\lambda_n^{-3})$$

$$\theta_n^{\boldsymbol\cdot}(t) = \varepsilon\lambda_n^{-2}\varphi_n'(\varkappa) + \varepsilon^2 O(I_{n,1}^*\lambda_n^{-2})$$

$$t \in [0, \Theta\varepsilon^{-1}], \quad n \geqslant 1$$

According to (3.5)-(3.8), we can derive expressions for the coordinates of the elastic displacements when $K \geqslant 2$. It follows from the formulae that the elastic deformations in the control process $f_{0,1}(\varkappa)$, $t \in [0, T]$, consist of quasistatic displacements and relatively small rapid oscillations. The principal term (with respect to ε) of the quasistatic deformations is

$$U_0(\varkappa, x) = \sum_{n=1}^{\infty} \frac{\varphi_n(\varkappa)}{\lambda_n^2}\xi_n(x), \quad U = U_0 + \varepsilon U_1 + \ldots + \varepsilon^K U_K \tag{3.10}$$

$$u(t, x, \varepsilon) = U(\varkappa, x, \varepsilon) + w(t, x, \varepsilon) \equiv U_0(\varkappa, x) + W(t, x, \varepsilon)$$

where w satisfies estimates of the type $O(\varepsilon^{K+1-\gamma})$ or $O(\varepsilon^{K+2-\gamma})$ (see (3.8)).

We will now study the relative vibrations of the system after the end of the control process $T < t \leqslant T_f$. Defining $\varphi_n(\varkappa) \equiv 0$ for $t > T$ and taking the terminal conditions (3.2) into account, we obtain the following estimates (see (3.1)):

$$|\theta_n(t)| = \varepsilon^{K+1}\lambda_n^{-(K+2)}O(I_{n,K}), \quad T \leqslant t \leqslant T_f \tag{3.11}$$

$$|\theta_n^{\boldsymbol\cdot}(t)| = \varepsilon^{K+1}\lambda_n^{-(K+1)}O(I_{n,K}^*), \quad T_f = \Theta_f\varepsilon^{-1}$$

Thus, the application of smooth and even compactly supported controls causes the centre of mass to change its state of motion to a considerable degree. There remain in the system relatively small elastic vibrations, whose amplitude may be estimated in advance and made negligibly small by suitable choice of the control. This possibility is of no little interest in practical situations demanding precision control of complex hybrid systems, since our conclusions are also valid for more general systems.

4. Concluding remarks. *4.1. Generalization of the control problem to two-mass or multi-mass hybrid systems.* Let us consider a certain generalization of the mechanical model of the system in Fig.1: imagine another mass m_l at the right end $(x = l)$, and denote the mass at the left end $(x = 0)$ by m_0. Non-dimensionalizing in accordance with (1.4), we have $m_{0,1}^* = m_{0,l}(\rho l)^{-1}$. We obtain a boundary-value problem of type (1.1):

$$u^{\cdot\cdot} = u'', \quad 0 < x < 1; \quad u = u(t, x) \tag{4.1}$$
$$m_0 u^{\cdot\cdot}(t, 0) = u'(t, 0) + f_0(t), \; m_1 u^{\cdot\cdot}(t, 1) = -u'(t, 1) + f_1(t)$$

If $f_{0,1}(t)$ are given, the initial conditions may have the general form (1.2). As before, we consider the control problem in its restricted formulation (see Sect.3, formulae (3.3)). The solution of problem (4.1) may be tackled along lines similar to those used in Sect.2. The selfadjoint eigenvalue and eigenfunction problem and its solution are as follows:

$$X'' + \lambda^2 X = 0, \quad X'(0) = -m_0\lambda^2 X(0), \quad X'(1) = -m_1\lambda^2 X(1) \tag{4.2}$$
$$D(m_0, m_1, \lambda) \equiv \lambda^2 [(1 - \lambda^2 m_0 m_1)\sin\lambda + \lambda(m_0 + m_1)\cos\lambda] = 0, \quad D(m_0, m_1, \lambda) \equiv D(m_1, m_0, \lambda)$$
$$\lambda = \arg D: \lambda_0 = 0, \quad \lambda_n = \lambda_n(m_0, m_1) \equiv \lambda_n(m_1, m_0)$$
$$X_0 = a_0\varphi_0 \equiv \text{const}, \quad X_n = a_n\varphi_n(x, m_0, m_1)$$
$$\varphi_0 \equiv 1, \; \varphi_n \equiv \cos\lambda_n x - m_0\lambda_n \sin\lambda_n x$$
$$\| X_n \|^2_\chi = a_n^2 \| \varphi_n \|_\chi^2, \quad \chi = \chi(x, m_0, m_1) = 1 + m_0\delta(x) + m_1\delta(x - 1)$$
$$\| \varphi_n \|_\chi^2 = r_n^2 = \int_0^1 \varphi_n^2(x, m_0, m_1)\chi(x, m_0, m_1)\,dx = {}^1/_2 + {}^1/_2 m_0 + {}^1/_2 m_0^2\lambda_n^2 +$$
$$m_1(\cos\lambda_n - m_0\lambda_n\sin\lambda_n)^2 + {}^1/_2\lambda_n(\lambda_n^{-2} - m_0^2)\sin\lambda_n\cos\lambda_n - m_0\sin\lambda_n, \quad n \geqslant 1$$
$$\| \varphi_0 \|_\chi^2 = M = 1 + m_0 + m_1, \quad (\varphi_n, \varphi_{n'})_\chi = r_n^2\delta_{nn'}$$

Let us briefly investigate the roots $\lambda_n(m_0, m_1)$ of the characteristic equation $D = 0$ (4.2). The equation and its roots are symmetric with respect to $m_0, m_1 \geqslant 0$. At $m_0 = 0$ or $m_1 = 0$ it is identical with (2.1). If $m_0 \to \infty\ (m_1 > 0)$ or $m_1 \to \infty\ (m_0 > 0)$, we obtain in the limit the characteristic equation $\operatorname{ctg}\lambda = m_{1,0}\lambda$, corresponding to a rigidly attached left or right end. But if both masses go to infinity, $m_{0,1} \to \infty$, we obtain vibrations of a bar attached at both ends: $\sin\lambda = 0, \lambda_n = \pi n$. In addition, if $m_0, m_1 > 0$ are finite, then as $n \to \infty$ the asymptotic form of λ_n also corresponds to a bar attached at both ends: $\lambda_n = \pi n + (m_0 + m_1)\cdot(m_0 m_1 \pi n)^{-1} + O(n^{-2})$. For small $m_0, m_1 \ll 1$ and "not too large n" $(1 \leqslant n \leqslant n^*)$ the asymptotic formula describes the case of a free rod:

$$\lambda_n = \pi n - \pi n(m_0 + m_1)(1 - m_0 - m_1) + O(((m_0 + m_1)\pi n)^3)$$

The eigenfunctions $\xi_n(x, m_0, m_1) = \varphi_n r_n^{-1}$ are orthonormal with weight $\chi = \chi(x, m_0, m_1)$ and have the basis property. Using the approach set out in Sect.2, we obtain a denumerable system of equations for the coordinates $\theta_0(t), \theta_n(t)\ (n \geqslant 1)$ similar to (2.3) where $m = m_0 + m_1$, $M = 1 + m$, φ_n and λ_n, r_n as defined in (4.2); the scalar product is defined relative to weight χ. The rest of the construction is similar to that of Sect.3.

We will now consider a multimass control system of $p + 1$ rigid bodies connected in series by $p, p \geqslant 1$, distributed elastic elements. It is assumed that $\rho_i, \sigma_i > 0$ are the characteristics of the distributed elements; $m_0, m_i\ (i = 1, 2, \ldots, p)$ are the masses of the bodies, some of which may be zero. As a result we obtain a solution of the system of simultaneous boundary-value problems

$$\rho_i u_i^{\cdot\cdot} = \sigma_i u_i'', \quad 0 < x_i < l_i; \quad u_i = u_i(t, x_i) \tag{4.3}$$
$$m_j u_j^{\cdot\cdot}(t, l_j) = \sigma_{j+1}u'_{j+1}(t, 0) - \sigma_j u_j'(t, l_j) + F_j(t), \quad j = 0, 1, \ldots, p;$$
$$\sigma_0 = \sigma_{p+1} = 0,$$
$$u_k(t, l_k) = u_{k+1}(t, 0), \quad k = 1, 2, \ldots, p - 1$$

Here $F_j(t)$ are the control functions applied to the masses $m_0, m_i\ (i = 1, \ldots, p)$. The corresponding system of eigenvalue and eigenfunction problems is obtained by separation of variables:

$$u_i = u_i(t, x_i) = \sum_{n=0}^{\infty} \theta_n(t) X_{i,n}(x_i) \tag{4.4}$$
$$X_i'' + \lambda^2 X_i = 0, \quad X_i = X_i(x_i), \quad 0 \leqslant x_i \leqslant l_i$$
$$X_i(x_i) = a_i\cos\lambda x_i + b_i\sin\lambda x_i$$
$$\lambda^2(m_j\sigma_j\rho_j^{-1})X_j(l_j) = \sigma_{j+1}X_{j+1}(0) - \sigma_j X_j'(l_j)$$
$$X_k(l_k) = X_{k+1}(0)$$

We obtain a system of $2p$ boundary-value problems for the $2p$ functions $X_i(x_i)$, containing $2p$ arbitrary constants a_i, b_i. For the corresponding homogeneous linear algebraic system to have a non-trivial solution, the parameter λ must take values satisfying the characeristic equation $D(m, \rho, \sigma; \lambda) = 0$, where D is the determinant of a $(2p \times 2p)$ matrix; m, ρ, σ are vectors of the parameters. As the expressions in question are rather cumbersome, it is not convenient to write out the determinant in its general form. However, we note that $D = \lambda^{p+1}D_p$, where D_p is a quasipolynomial containing powers of λ and products of the functions $\sin \lambda l_i$, $\cos \lambda l_i$. The zero root $\lambda_0 = 0$ corresponds to motion of the centre of mass of a system with mass $M = m_0 + \Sigma_i (m_i + \rho_i l_i)$ under the action of a total force $F(t) = F_0(t) + \Sigma_i F_i(t)$. The rest of the construction proceeds along lines analogous to those described above. The smooth even controls $f_j = f_j(\varkappa)$ $(j = 0, 1, \ldots, p)$ ensure the required relative accuracy $O(\varepsilon^{K+1-\gamma})$ of the control in motion of a hybrid vibrating system of more general form (4.3).

A similar procedure can also be adopted for controllable systems with inhomogeneous characteristics $\rho_i = \rho_i(x_i)$, $\sigma_i = \sigma_i(x_i)$, which may be investigated either accurately or approximately, e.g., by the technique of /12/.

4.2. Analysis of motion for some specific control strategies. Let us consider the motion of a hybrid system of type (1.1) or (4.1) for some specific choices of the controls $f_0(\varkappa)$, $f_1(\varkappa)$. We first consider on-off controls:

$$f_{0,1}(\varkappa) = a_{0,1} \operatorname{sign}(\alpha_{0,1} - \beta_{0,1}\varkappa), \quad \varkappa \in [0, \theta]$$

where the parameters $a_{0,1}, \alpha_{0,1}, \beta_{0,1} = \text{const}$ satisfy conditions (3.4). Since $f_{0,1}(\varkappa)$ are not in class Φ^0, we have no estimates of type (3.9) for the elastic vibrations. It can be shown by direct integration that the vibrations and "quasistatic" displacement are of the same order of magnitude $|a_{0,1}|$ both during the control process $\varkappa \in [0, \theta]$ and thereafter $\varkappa > \theta$. Moreover, depending on the switching time $\varkappa_{0,1}^* = \alpha_{0,1}\beta_{0,1}^{-1}$ and the terminal time $\varkappa = \theta$, the amplitude of the vibrations may increase. If $a_0, a_1 \sim \varepsilon$, i.e., $s^0 \sim \varepsilon^{-1}$, $v^0 \sim 1$, the amplitude of the residual vibrations will also be of the order of ε.

We will now consider trapezoidal-shaped controls $f_0(\varkappa)$ and $f_1(\varkappa)$ in class Φ^0 with a finite angle of inclination. By (3.5) and (3.8), we obtain $\theta_n(t) = \varphi_n(\varkappa) + \varepsilon O(\varphi_n'\lambda_n^{-3})$, $\theta_n^{\cdot}(t) = \varepsilon O(\varphi_n'\lambda_n^{-2})$ for $t \in [0, T]$ and $\theta_n(t) = \varepsilon O(\varphi_n'\lambda_n^{-3})$, $\theta_n^{\cdot}(t) = \varepsilon O(\varphi_n'\lambda_n^{-2})$ for $t \geqslant T$. If $f_{0,1} \sim \varepsilon$, $f_{0,1}' \sim \varepsilon$, the absolute deviation from the quasistatic approximation is $O(\varepsilon^2)$.

Possible mechanical models of the hybrid controllable systems just considered are uniaxial series of elastic shafts and absolutely rigid flywheels, cords connected in series with material points at their ends, etc.

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